# Lecture 16

## 1 Angular momentum, cont.

**Note:** tower cannot be infinite!

$$\hat{L}^2 = L_x^2 + L_y^2 + L_z^2 \geq L_z^2 \tag{1}$$

There must exist a maximum allowed $m$ given $l$! ie there must exist $m_+$ st

$$L_+ Y_{lm} = 0 \tag{2}$$

in the same way that, in the harmonic oscillator, $a\psi_0 = 0$. What is $m_+(l)$?

**Note:** If $L_+ Y_{lm} = 0$, then $\langle L_+ Y_{lm} | L_+ Y_{lm} \rangle = 0$

$$\langle L_+ Y_{lm} | L_+ Y_{lm} \rangle = \int d\theta d\phi (L_+ Y_{lm})^* L_+ Y_{lm} \tag{3}$$

$$= \int d\theta d\phi Y_{lm}^* (L_+)^\dagger \, L_+ \, Y_{lm} \tag{4}$$

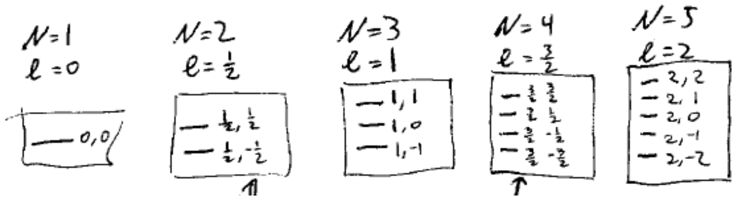
$$= \int d\theta d\phi Y_{lm_+}^* L_- \, L_+ \, Y_{lm_+} \tag{5}$$

$$= \int d\theta d\phi Y_{lm_+}^* (L^2 - L_z^2 - \hbar L_z) Y_{lm_+} \tag{6}$$

$$= \int d\theta d\phi Y_{lm}^* \hbar^2 (l(l+1) - m_+^2 - m_+) Y_{lm_+} \tag{7}$$

$$= \hbar^2 [l(l+1) - m_+(m_+ + 1)] \langle Y_{lm_+} | Y_{lm_+} \rangle = 0 \tag{8}$$

so $l = m_+$ and, similarly with $L_-$ and $Y_{lm_-}$, $-l = m_-$. Thus our ladders have rungs $m = -l$ to $m = l$, with $-l+1$, $l-1$, etc in between. There are N states: $2l = N-1$ so $l = \frac{N-1}{2}$. $l$ is therefore either an integer (odd number of states N) or a half-integer (even number of states N).




etc.

**So:** Eigenfunctions of $\hat{L}^2, \hat{L}_z$ are $Y_{lm}$ st

$$\hat{L}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm} \tag{9}$$

$$L_z Y_{lm} = \hbar m Y_{lm} \tag{10}$$

So $m$ ranges from $-l$ to $l$ and $l = \frac{N-1}{2}$. This is like saying $\hat{N}\phi_n = n\phi_n$ where $n$ is greater than or equal to zero.

How should we visualize this?

1. First, $m = \pm l$ not pointing in $\hat{z}$!

$$\langle L^2 \rangle = \hbar^2 l(l+1) \tag{11}$$

$$\langle L_z^2 \rangle = \hbar^2 m^2 = \hbar^2 l^2 \tag{12}$$

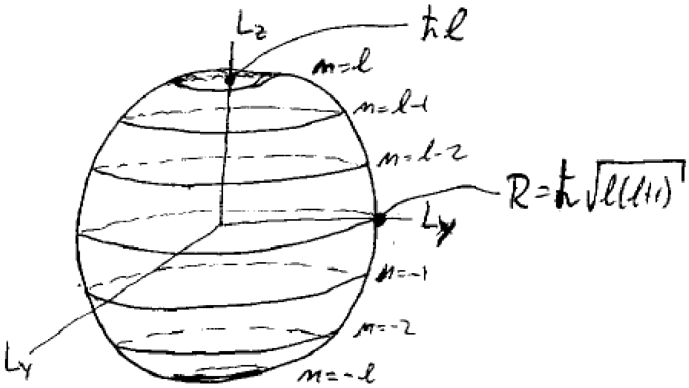
   so, $\hbar\sqrt{l}$ in $L_x, L_y$

2. Easy case: $\langle L_x^2 \rangle = \langle L_y^2 \rangle = \hbar^2 \frac{l}{2}$ so

$$\Delta L_x \Delta L_y = \frac{\hbar^2 l}{2} = \frac{1}{2} \langle [L_x, L_y] \rangle = \frac{\hbar}{2} \langle L_z \rangle = \frac{\hbar^2 l}{2} \tag{13}$$

3. State with definite $\hat{L}^2$, $L_z$ does **not** have definite $L_x$, $L_y$.

4. Plot on computer!


L_z
ħl
m=l
m=l-1
m=l-2
R=ħ√l(l+1)
L_y
m=-1
m=-2
m=-l
L_x